COMAP

R4R

Refresh for Relevance

WWW.COMAP.ORG/R4R

The Consortium for Mathematics and Its Applications (COMAP)
175 Middlesex Turnpike, Suite 3B
Bedford, MA 01730

Published and distributed by

COMAP, Inc.
www.comap.org

2025 Disclaimer

This reprinted publication contains some information that may be outdated or no longer accurate, such as web links and contact details. For the most up-to-date information about COMAP and its publications, please visit: www.comap.org.

The text of this publication, or any part thereof, is made available as part of the Refresh for Relevance (R4R) project. More information about the project can be found at

www.comap.org/R4R

UMAP

Modules in
Undergraduate
Mathematics
and Its
Applications

Published in
cooperation with

The Society for
Industrial and
Applied Mathematics,

The Mathematical
Association of America,

The National Council
of Teachers of
Mathematics,

The American
Mathematical
Association of
Two-Year Colleges,

The Institute for
Operations Research
and the Management
Sciences, and

The American
Statistical Association.

COMAP

Module 766

Using Original Sources to Teach the Logistic Equation

Bonnie Schulman

Applications of Mathematical
Modeling, Biology to Differential
Equations and Calculus

COMAP, Inc., Suite 210, 57 Bedford Street, Lexington, MA 02420 (781) 862–7878

INTERMODULAR DESCRIPTION SHEET:	UMAP Unit 766
TITLE:	Using Original Sources to Teach the Logistic Equation
AUTHOR:	Bonnie Shulman Dept. of Mathematics Bates College Lewiston, ME 04240 `bshulman@@abacus.bates.edu`
MATHEMATICAL FIELD:	Differential equations, calculus
APPLICATION FIELD:	Mathematical modeling, biology
TARGET AUDIENCE:	Students in a course in differential equations or second-semester calculus.
ABSTRACT:	This Module uses original data, diagrams, and texts from three original sources to develop the logistic model of growth in natural systems with limited resources. The logistic differential equation and the familiar S-shaped logistic curve have applications in solving problems in ecology, biology, chemistry, and economics. The Module illustrates with concrete examples how mathematics develops, and it provides insights into the assumptions that drive the modeling process.
PREREQUISITES:	The reader is assumed to be familiar with geometric and arithmetic progressions. From calculus: differentiation and integration of elementary functions. A basic introduction to differential equations is desirable, but the Module itself might serve as just such an introduction.

COMAP, Inc., Suite 210, 57 Bedford Street, Lexington, MA 02173
(800) 77-COMAP = (800) 772-6627, or (617) 862-7878; `http://www.comap.com`

Using Original Sources to Teach the Logistic Equation

Bonnie Shulman
Dept. of Mathematics
Bates College
Lewiston, ME 04240
bshulman@@abacus.bates.edu

Table of Contents

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS (UMAP) PROJECT

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications, to be used to supplement existing courses and from which complete courses may eventually be built.

The Project was guided by a National Advisory Board of mathematicians, scientists, and educators. UMAP was funded by a grant from the National Science Foundation and now is supported by the Consortium for Mathematics and Its Applications (COMAP), Inc., a nonprofit corporation engaged in research and development in mathematics education.

Paul J. Campbell	Editor
Solomon Garfunkel	Executive Director, COMAP

1. Introduction

There is a common perception of mathematics as a finished product invented by dead geniuses. In an effort to dispel this notion and convey the excitement of mathematics as a living, breathing, and growing body of knowledge, created by human beings very much like ourselves, I have turned to original sources. This Module has grown over time as I integrated material from the three original papers into courses in differential equations, modeling, and even introductory calculus:

- The first paper is an oft-cited classic by Pearl and Reed [1920], who are usually credited with being the first to use the logistic equation to describe the growth of the population of the United States.
- The next paper is the text of a presidential address to the Royal Statistical Society in England, by G. Udny Yule [1925], which contains an excellent critical history of the logistic model and summarizes the work of Pearl and Reed, as well as that of Verhulst.
- From Yule, I learned that Pierre-François Verhulst, a Belgian sociologist and mathematician, was actually the first to propose and publish a formula for the law of growth for a population confined to a specified area [Verhulst 1845].

Yule states, "[p]robably owing to the fact that Verhulst was greatly in advance of his time, and that the then existing data were quite inadequate to form any effective test of his views, his memoirs fell into oblivion" [Yule 1925, 4]. Apparently, some 80 years later, Pearl and Reed had arrived independently at the same result. Verhulst's work did eventually come to their attention; in fact, Yule acknowledges [Yule 1925, 5] that he is indebted to Pearl's book [Pearl 1922] for the references to Verhulst. Verhulst wrote in French; but with dictionary in hand, and a rudimentary high-school background in the language (like my own), the text is quite comprehensible.

This Module uses original data, diagrams, and text from these three original sources. The numbering of equations and figures follows that in the original, so it is not consistent throughout the Module. Also, one should be alert to changes in notation (population is represented as p or y, time as t or x, respectively). Some of the notation may also be confusing if the text is not read carefully; for example, p' and y' represent particular values of p and y, *not* derivatives. It is assumed that the audience understands and can work with geometric and arithmetic progressions and has had a basic introduction to differential equations. Quotations from original sources in English are either placed between quotation marks or else set off as displays with indented margins. Sources in French are rendered in split-page format, with the original French on the left and my translation on the right.

The exercises are designed to stimulate thought and inculcate the habit of reading mathematics with a pencil in hand, always ready to verify and check all claims made, and work out the equations for oneself.

This Module is intended to illustrate how mathematical knowledge grows—by fits and starts, rather than in a simple "linear" progression (as it is often presented in textbooks). Reading original sources, one notices that ideas are rediscovered and how later researchers borrow from and reinterpret the work of earlier mathematicians. Thus, in this Module, the same equations sometimes appear in slightly different forms, as they are reworked by various authors. The reader is encouraged to use these examples of what may at first appear to be redundancies in the text, as opportunities to compare and contrast different points of view, which can lead to further insights into the mathematics as well as its historical development.

2. The Logistic Equation

The logistic equation is used to model natural systems, involving growth with limited resources. This simple function, along with the differential equation that it satisfies and its familiar S-shaped curve, is ubiquitous and familiar to mathematicians and natural and social scientists alike. In the excerpts that follow, one can trace the early history of this model and gain insight into the assumptions on which it is based.

2.1 Yule's Summary of Malthus's Argument

We begin with Yule's summary of the history of attempts to model populations.

> Malthus, as will be well remembered by anyone who has ever read the *Essay on the Principle of Population*, reaches his conclusions by a *reductio ad absurdum* argument—the argument, to put it briefly, that if the population of a confined area increases without limit in geometric progression there will soon be millions without any food. [Yule 1925, 2]

Exercises

1. Look up Malthus's essay [1798], which has often been reprinted. Write a short summary of the key points.

2. Explain what is meant by a reductio ad absurdum argument.

> And Malthus seems almost to enjoy the depicting of horrors (or horrours, if one may use the earlier spelling, which in some odd way seems to add enormously to the effect) . . . . Malthus assumes that the population will double every 25 years, while the produce will be doubled in the first 25

years, but after that will only continue to increase in arithmetic progressions (*Essay* (1798) pp. 56-8). "And at the conclusion of the first century the population would be 112 millions, and the means of subsistence only equal to the support of 35 millions, which would leave a population of 77 millions totally unprovided for." It is a shocking picture, and it leaves our feelings so harrowed as to be capable of little further sympathy with the plight of the world, in which "in two centuries and a quarter the population would be to the means of subsistence as 512 to 10."

[Yule 1925, 3]

Exercises

3. If a population doubles every 25 years, and is 112 million at the end of one hundred years, what was the initial population?

4. Given the initial population above, with the same doubling time of 25 years, what will the population be in 225 years?

5. Using the fact that "the population is to the means of subsistence as 512 to 10," calculate how many millions can be supported after 225 years.

6. If the means of subsistence grows arithmetically, how many more people can be supported every 25 years? (Hint: use your knowledge of how many people can be supported after 225 years, and the fact that 35 million people are supported at the end of one hundred years. Also recall that the produce doubled in the first 25 years.)

7. Suppose two bacteria are placed in a Petri dish with a fixed amount of space. At the end of one minute, the number of bacteria has doubled (that is, there are now four bacteria in the dish). If there is exactly enough space in the dish for 1024 bacteria, how long before the space runs out, if

a) the number of bacteria increases in a geometric progression (this is called *exponential growth*);

b) the number of bacteria increases in an arithmetic progression (this is called *linear growth*)?

But there is another and more serious disadvantage attaching to such a mode of argument; it tells us very little. The only conclusion that can be drawn is that a population, confined to a specified area, does *not* increase in geometric progression. As to the true form of the law of increase, the argument gives us no information. [Yule 1925, 3]

2.2 Verhulst's Argument

Verhulst, a Belgian, published his papers in French. Below is the original text, and a rough translation, of some of his memoir. Yule refers to this memoir later in his address.

Au nombre des causes qui exercent une action constante sur l'accroissement de la population, nous placerons la fécondité propre a l'espèce humaine, la salubrité du pays, les mœurs de la nation que l'on considère, ses lois civiles et religieuses. Quant aux causes variable que l'on ne peut pas regarder comme les accidentelles, elles se résument généralement dans la difficulté de plus en plus grande que la population éprouve à se procurer des subsistances, lorsqu'elle est devenue assez nombreuse pour que toutes les bonnes terres se trouvent occupées.

Quand on ne tient pas compte de la difficulté dont nous venons de parler, il faut admettre qu'en vertu des causes constantes, la population doit croître en progression géométrique. En effet, si 1000 âmes sont devenue 2000 au bout de 25 ans, par exemple, il n'ya pas de raison pour que ces 2000 ne deviennent pas 4000 au bout de 25 années suivantes. [Verhulst 1845, 4]

The causes that exert a constant effect on the growth of population are: fertility, the wealth of the country, the death rate, and the nation's civil and religious laws. As for variable causes that aren't accidental, we must consider the difficulty in finding resources when the population becomes too numerous and all the good land is occupied.

If we consider only the constant causes, the population must grow in a geometric progression. In other words, if 1000 people become 2000 in 25 years, for example, there is no reason that 2000 should not become 4000 in the following 25 years.

Exercise

8. Does Verhulst's argument make sense to you? His is the first attempt to model population growth quantitatively. He tries to capture our commonsense notion of how populations grow, when such factors as birth and death rate are constant. Think about rabbits. If you start with 10 and the population doubles in 25 days, does it seem reasonable that if you started with 20, you would have 40 rabbits in 25 days? In what situations would this *not* be a reasonable assumption?

Les États Unis nous offrent un exemple de cette grande vitesse d'accroissement de la population. On y comptait, d'après le recensements officiels, [Verhulst 1845, 4]	The United States [in the late eighteenth and early nineteenth centuries] offers just such an example of a rapidly growing population that is expanding as if it had unlimited resources. A list of the official census figures follows.

En 1790	....	3,929,827	âmes [souls]
1800	....	5,305,925	
1810	....	7,239,814	
1820	....	9,638,151	
1830	....	12,866,020	
1840	....	17,062,566	

[Verhulst 1845, 4]

Si l'on prend pour la population de 1795 le chiffre 4,617,876, moyen entre celui de 1790 et celui de 1800, et qu'on fasse de même pour les années 1805, 1815, 1825 and 1835, on pourra évaluer approximativement les progrès de la population de 5 en 5 ans. C'est ainsi que nous avons formé le tableau suivant, dans lequel nous avons arrondi les chiffres et désigné par r le rapport de chaque population à celle qui la précède de 25 ans: [Verhulst 1845, 5]	In the following table, we take these official census figures for decades, and approximate the population in inter-censal years using the arithmetic mean. The third column lists the ratio, r, of each population to that of the preceding 25 years. The numbers are rounded.

This table (on the next page) illustrates a defining characteristic of exponential growth: for equal increments of time (in this case, 25-year intervals), the ratio between succeeding populations is constant (in this case about 2.1).

ANNÉES.	POPULATION.	VALEURS DE r.
1700.	3,930,000	
1795.	4,618,000	
1800.	5,306,000	
1805.	6,273,000	
1810.	7,240,000	
1815.	8,439,000	2.147
1820.	9,638,000	2.087
1825.	11,252,000	2.120
1830.	12,866,000	2.052
1835.	14,964,000	2.076
1840.	17,063,000	2.021

[Verhulst 1845, 5]

Nous n'insisterons pas davantage sur l'hypothèse de la progression géométrique, attendu qu'elle ne se réalise que dans des circonstances tout à fait exceptionnelles; par exemple, quand un territoire fertile et d'une étendue en quelque sorte illimitée, se trouve habité par un peuple d'une civilisation très-avancé, comme celle des premiers colons des États-Unis. [Verhulst 1845, 6]

[Unlike Malthus,] We readily admit that the hypothesis that populations increase in geometric progression is valid only in exceptional circumstances, as for example when a fertile and vast territory is inhabited by a technologically advanced people, like the early colonists in the United States.

2.3 Pearl and Reed's Data and Methods

Here is the opening paragraph of Pearl and Reed's paper.

It is obviously possible in any country or community of reasonable size to determine an empirical equation, by ordinary methods of curve fitting, which will describe the normal rate of population growth. Such a determination will not necessarily give any inkling whatever as to the underlying organic laws of population growth in a particular community. It will simply give a rather exact empirical statement of the nature of the changes which have occurred in the past. No process of empirically graduating raw data with a curve can in and of itself demonstrate the fundamental law which causes the occurring change. In spite of the fact that such mathematical expressions of population growth are purely empirical, they have a distinct and considerable usefulness. This usefulness arises out of the fact that actual counts of population by census methods are made at only relatively infrequent intervals, usually 10 years and practically never oftener than 5 years. For many statistical purposes, it is necessary to have as accurate an estimate as possible of the population in inter-censal years. This applies not only to the years following that on which the last census was taken, but also to the inter-censal years lying between prior censuses. For purposes of practical statistics it is highly important to have these inter-censal estimates of population as accurate as possible, particularly for the use of the vital statistician, who must have these figures for the calculation of annual death rates, birth rates and the like. [Pearl and Reed 1920, 275]

TABLE I

SHOWING THE DATES OF THE TAKING OF THE CENSUS AND THE RECORDED POPULATIONS FROM 1790 TO 1910

DATE OF CENSUS		RECORDED POPULATION (REVISED FIGURES FROM STATISTICAL ABST., 1918)
Year	Month and Day	
1790	First Monday in August	3,929,214
1800	First Monday in August	5,308,483
1810	First Monday in August	7,239,881
1820	First Monday in August	9,638,453
1830	June 1	12,866,020
1840	June 1	17,069,453
1850	June 1	23,191,876
1860	June 1	31,443,321
1870	June 1	38,558,371
1880	June 1	50,155,783
1890	June 1	62,947,714
1900	June 1	75,994,575
1910	April 15	91,972,266

Table 1 from Pearl and Reed [1920, 277].

Exercise

9. Consider the data from Pearl and Reed's **Table 1**. After rounding up or down to the nearest thousand, estimate the population for the inter-censal years, assuming that the population is increasing from 1790 to 1910 in

a) a geometric progression;

b) an arithmetic progression.

Which estimate do you think is better? Why? How might you improve your estimate?

> The usual method followed by census offices in determining the population in inter-censal years is one or the other of two sorts, namely, by arithmetic progression or geometric progression. These methods assume that for any given short period of time the population is increasing either in arithmetic or geometric ratio. Neither of these assumptions is ever absolutely accurate even for short intervals of time, and both are grossly inaccurate for the United States, at least, for any considerable period of time. What actually happens is that following any census estimates are made by one of another of these methods of the population for each year up to the next census, on the basis of data given by the last two censuses only. When that next census has been made, the previous estimates of the inter-censal years are corrected and adjusted on the basis of the facts brought out at that census period. [Pearl and Reed 1920, 275–276]

Exercises

10. Given the data in **Table 1**, how would you determine an empirical equation that fits the data? Do not actually find such an equation, just explain how you would go about it.

11. What is the difference between finding an equation of "best fit" for a given set of data, and determining a "fundamental law" that "causes the occurring change"?[1]

We continue the quotation from Pearl and Reed's paper:

> It would be the height of presumption to attempt to predict *accurately* the population a thousand years hence. But any real law of population growth ought to give some general and approximate indication of the number of people who would be living at that time within the present area of the United States, provided no cataclysmic alteration of circumstances has in the meantime intervened.
>
> It has seemed worth while to attempt to develop such a law, first by formulating a hypothesis which rigorously meets the logical requirements, and then by seeing whether in fact the hypothesis fits the known facts. The general biological hypothesis which we shall here test embodies as an essential feature the idea that the rate of population increase in a limited area at any instant of time is proportional (a) to the magnitude of the

[1]Instructors may want to discuss the work of Brahe, Kepler, and Newton in this context, or ask students to research this. Brahe was an observer who collected the most accurate data of his time on the motions of the planets. Kepler discerned patterns in the data and derived equations to describe the paths planets followed (ellipses) and relationships between a planet's period of revolution and its distance from the sun. Newton explained the observations through a general law (of gravitation) that implied Kepler's equations and much more. Kepler used simple induction to express a regularity of nature, while Newton may be said to have discovered a fundamental causal relationship. See, for example, Kuhn [1970, 209–219] and Abers and Kennel [1977, 105–132].

population existing at that instant (amount of increase already attained) and (b) to the still unutilized potentialities of population support existing in the limited area. [Pearl and Reed 1920, 281]

Exercise

12. Let y represent the population at time x. Write an equation for the relationship between dy/dx (the rate of population increase) and the population that models the above hypotheses.

The following conditions should be fulfilled by any equation which is to describe adequately the growth of population in an area of fixed limits.

1. Asymptotic to a line $y = k$ when $x = +\infty$.
2. Asymptotic to a line $y = 0$ when $x = -\infty$.
3. A point of inflection at some point $x = \alpha$ and $y = \beta$.
4. Concave upwards to left of $x = \alpha$ and concave downward to right of $x = \alpha$.
5. No horizontal slope except at $x = \pm\infty$.
6. Values of y varying continuously from 0 to k as x varies from $-\infty$ to $+\infty$.

In these expressions y denotes population, and x denotes time. [Pearl and Reed 1920, 281]

Exercise

13. Give reasons why "any equation which is to describe adequately the growth of a population in an area of fixed limits" should satisfy each of the six conditions listed.

a) Draw a graph that illustrates each of the conditions separately.

b) Draw one graph that meets all of the conditions simultaneously.

An equation which fulfills these requirements is

$$y = \frac{be^{ax}}{1 + ce^{ax}} \tag{ix}$$

when a, b and c have positive values. [Pearl and Reed 1920, 281]

Exercise

14. Verify that equation **(ix)** of Pearl and Reed meets each of the conditions **(1–6)**.

In this equation the following relations hold:

$$x = +\infty \qquad y = b/c \tag{x}$$

$$x = -\infty \qquad y = 0 \tag{xi}$$

Relations (x) and (xi) define the asymptotes. The point of inflection is given by $1 - ce^{ax} = 0$, or

$$x = -(1/a)\log c \qquad y = b/2c \tag{xii}$$

The slope at the point of inflection is $ab/4c$.

[Pearl and Reed 1920, 281–282]

Exercise

15. Verify the relations in **(x)**, **(xi)**, and **(xii)**.

Expressing the first derivative of (ix) in terms of y, we have

$$\frac{dy}{dx} = \frac{ay(b - cy)}{b} \tag{xiii}$$

[Pearl and Reed 1920, 282]

Exercise

16. Compare this equation with the one that you came up with above, in **Exercise 12**, to model Pearl and Reed's hypotheses. Show that if one lets $L = b/c$, then the equation above can be written in terms of only two constants, a and L. The third constant in **(ix)** arises as a constant of integration and depends on initial conditions.

The general form of the curve (ix) is shown in figure 2.

Putting the equation in this form shows at once that it is identical with that describing an autocatalyzed chemical reaction, a point to which we shall return later. [Pearl and Reed 1920, 282]

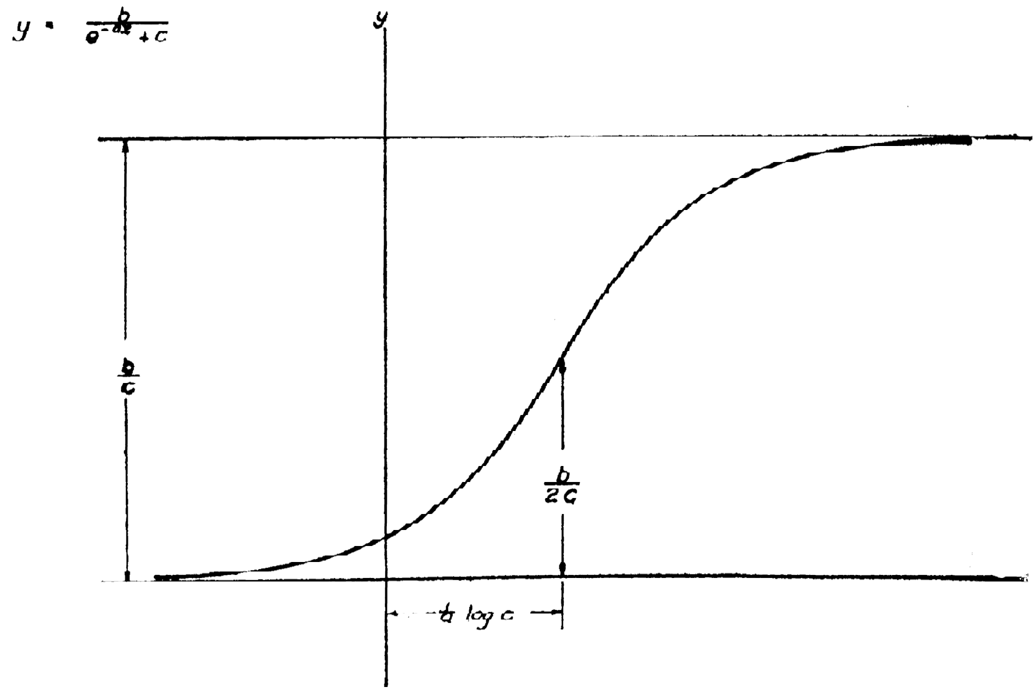

FIG. 2
General form of curve given by equation (ix).

Figure 2 from Pearl and Reed [1920, 282].

Exercise

17. Look up the definition of *autocatalysis*. In what ways is this process similar to population growth in an area of limited resources?

There is much that appeals to the reason in the hypothesis that growth of population is fundamentally a phenomenon like autocatalysis. In a new and thinly populated country the population already existing there, being impressed with the apparently boundless opportunities, tends to reproduce freely, to urge friends to come from older countries, and by the example of their well-being, actual or potential, to induce strangers to immigrate. As the population becomes more dense and passes into a phase where the still unutilized potentialties of subsistence, measured in terms of population, are measurably smaller than those which have already been utilized, all of these forces tending to the increase of population will become reduced. [Pearl and Reed 1920, 287]

2.4 Continuation of Yule's Account

We now return to Yule's historical account.

> Verhulst, Professor of Mathematics at the École Militaire, . . . states [in a memoir from 1838] that he had long since attempted to determine the probable form of the law of population, but had abandoned the investigation on account of the inadequacy of the data. But, as the course he had followed would, as it seemed to him, necessarily lead to the true law when sufficient data were available, and as the results at which he had arrived were of some interest, he had consented to M. Quetelet's invitation to publish them. Let p denote the population, t the time; then if the population is increasing in geometric progression
>
> $$dp/dt = mp.$$
>
> [Yule 1925, 43]

Exercise

18. What is the solution to this differential equation? Does this clarify the connection between "growing exponentially" and "growing in a geometric progression"? Explain.

> But since the rate of growth of the population is retarded by the increased number of the inhabitants, we must subtract from mp some unknown function of p, so that the differential equation to be integrated takes the form
>
> $$dp/dt = mp - \varphi(p).$$
>
> The simplest assumption that can be made as to the form of $\varphi(p)$ is to suppose $\varphi(p) = np^2$, which gives as the solution
>
> $$p = \frac{mp'e^{mt}}{np'e^{mt} + m - np'}. \qquad (*)$$
>
> where p' is the population at zero time, and the limiting population when t is infinite is m/n. [Yule 1925, 43]

Exercises

19. Verify that the function given for p above in **(*)** does satisfy the differential equation $dp/dt = mp - np^2$. Then show that as $t \to \infty$, $p \to m/n$.

20. Explain how $\varphi(p) = np^2$ is the "simplest" assumption that can be made about the form of $\varphi(p)$.

> Verhulst returns to the subject in a much longer memoir a few years later. [This is the memoir quoted in this Module.] The argument is here developed on slightly different and simpler lines. The freely-expanding population, it is admitted, must increase in geometric progression, the data for the U.S.A. 1790–1840 being used to illustrate the point. But suppose that the population has expanded up to the point when "the difficulty of finding good land has begun to make itself felt." Let the population at this epoch, which will be taken as zero time, be b: b is termed by Verhulst the "normal population." The "retarding function" now comes into play, and the differential equation may be written
>
> $$\frac{1}{p}\frac{dp}{dt} = l - f(p - b).$$
>
> (The retarding function is now, more naturally, taken as a retarding function for the logarithmic differential instead of dp/dt.)
>
> [Yule 1925, 43–44]

Exercises

21. What is a logarithmic differential?

22. Explain, in your own words, the role of the "retarding function" in this model of population growth.

> Only two conditions are necessary for the retarding function in its new form: it must increase indefinitely with the population, and it must vanish when $p = b$. [Yule 1925, 44]

Exercise

23. Justify and explain why these two conditions are necessary for the "retarding" function.

> The simplest form to assume is $n(p - b)$: we then have: – –
>
> $$\frac{1}{p}\frac{dp}{dt} = l - n(p - b),$$
>
> or, writing for brevity $m = l + nb$,
>
> $$\frac{1}{p}\frac{dp}{dt} = m - np.$$
>
> Verhulst now christens the curve a "logistic." He develops the principal properties, pointing out that the curve is symmetrical with respect to the point of inflection, and that the ordinate at the point of inflection is half the limiting ordinate. [Yule 1925, 44]

2.5 Return to Verhulst's Original Account

Here is the relevant passage from Verhulst.

Désignons par p la population, par t le temps, et par k et l des constantes indéterminées: si la population croît en progression géométrique pendant que le temps croît en progression arithmétique, on aura entre ces deux quantités la relation,

Let p be the population and t stand for time, with k and l undetermined constants. If the population grows in a geometric progression, while time grows in arithmetic progression, the two quantities will be related in the following way:

$$p = k \cdot 10^{lt}.$$

[Verhulst 1845, 5]

Exercise

24. Compare this to the solution that you got in **Exercise 18** above.

Soit p' une population correspondante à un temps t': il viendra

If p' is the population corresponding to time t', then this becomes

$$p = p' \cdot 10^{l(t-t')},$$

et si l'on appelle π la population existante au moment d'où l'on commence à compter le temps, l'équation précédente devient

and if one lets π be the population at the time one starts counting, the preceding equation becomes

$$p = \pi \cdot 10^{lt}. \quad \textbf{(1)}$$

... La période malthusienne de 25 ans suppose que p devient $2p$ quant t devient $t+25$, l'année étant prise pour unité de temps: on a donc le équation (...):

The "malthusian period" of 25 years assumes that p becomes $2p$ when t becomes $t + 25$, the year being taken as the unit of time. One then finds that

$$l = (1/25) \log 2 = .012041200.$$

[Verhulst 1845, 5–6]

Exercise

25. Verify each of the above equations. (Note that here $\log 2$ denotes a logarithm to the base 10. We will use $\ln 2$ to denote natural logarithms.)

La différentiation de l'équation (1) donne	Differentiating equation (1) gives

$$\frac{M}{p}\frac{dp}{dt} = l, \qquad \textbf{(2)}$$

. . . et en désignant par M le module par lequel il faut multiplier les logarithmes népériens pour les convertir en logarithmes vulgaires. [Verhulst 1845, 6]	where $M = \log e$.

Exercise

26. Show that if $M \ln x = \log x$, then $M = \log e$, for any real number x.

Cette quantité étant constante, on peut la prendre pour mesure de l'énergie avec laquelle la population tend à se développer, lorsqu'elle n'est point retenue par la crainte de manquer de subsistances. On a aussi, avec une exactitude d'autant plus grande que Δp et Δt sont plus petits,	The ratio of the rate of change of the population to the population itself is thus constant, and one can take this constant to be a measure of the energy with which the population tends to grow, when not constrained by limited resources. In fact, for small changes in p and t (Δp and Δt), we may say

$$M\Delta p = lp\Delta t;$$

et, si l'on prend pour Δt l'intervalle d'une année,	and if one takes Δt to be one year, we arrive at

$$\frac{\Delta p}{p} = \frac{l}{M},$$

c'est-à-dire que, dans le cas de la progression géométrique, l'excès annuel des naissances sur les décès, divisé par la population qui l'a fourni, donne un quotient constant . . . . C'est un fait d'observation que, dans toute l'Europe, le rapport de l'excès annuel des naissances sure les décès à la population qui l'a fourni, et par conséquent le coefficient l/M, va sans cesse en s'affaiblissant: de manière que l'accroissement annuel, dont la valeur absolue augmente continuellement lorsqu'il y a progression géométrique, paraît suivre un progression tout au plus arithmétique. Cette remarque confirme le célèbre aphorisme de Malthus, que *la population tend à croître en progression géométrique, tandis que la production des subsistances suit une progression tout au plus arithmétique*. [Verhulst 1845, 7]

which is to say that in the case of the population growing in geometric progression, the excess of annual births over deaths, divided by the population, is a constant ratio. However, throughout Europe, it is observed that this ratio, l/M, in fact decreases. This observation confirms the celebrated aphorism of Malthus, that *the population tends to grow in geometric progression while the production of food follows a more or less arithmetic progression*.

As you read Verhulst's original derivation below of the logarithmic differential equation, compare it to Yule's treatment above (on p. 13, following **Exercise 23**).

On peut faire un infinité d'hypothèses sur la loi d'affaiblissement du coefficient l/M. La plus simple consiste à regarder cet affaiblissement comme proportionnel à l'accroissement de la population, depuis le moment où la difficulté de trouver de bonnes terres a commencé à se faire sentir. Nous appellerons *population normale*, et nous désignerons par b, celle qui correspond à cette époque remarquable, à partir de laquelle nous compterons le temps: puis, ayant dénoté par n un coefficient indéterminé, nous remplacerons l'équation différentielle

One could make an infinite number of hypotheses about the law of decrease of the coefficient l/M. The simplest is to consider the decrease to be proportional to the growth of the population, from the time when the difficulty of finding good land begins to be felt. We will begin counting from this time, and call the population at this time the *normal population*, designated by b. Then, letting n denote an undetermined coefficient, we replace the differential equation

$$\frac{M}{p}\frac{dp}{dt} = l,$$

relative à la progression géométrique, par

by

$$\frac{M}{p}\frac{dp}{dt} = l - n(p - b),$$

d'où, en posant, pour abréger, $m = l + nb$,

and substituting $m = l + nb$,

$$\frac{M}{p}\frac{dp}{dt} = m - np,$$

et

and

$$dt = \frac{M\,dp}{mp - np^2}.$$

Cette équation étant intégrée donne, en observant que $t = 0$ répond à $p = b$,

We integrate this equation, noticing that $t = 0$ corresponds to $p = b$,

$$t = \frac{1}{m}\log\frac{p(m - nb)}{b(m - np)}.$$

Nous donnerons le nom de *logistique* à la courbe caractérisée par l'équation précédente. [Verhulst 1845, 8–9]

and give the name *logistic* to the curve characterized by the previous equation.

How exciting! Here is where Verhulst first "christens" the equation a logistic. Why? In its modern incarnation, the logistic equation is usually written with population expressed as a function of time (population as the dependent variable). This perhaps more familiar form of the equation involves an exponential. Verhulst wrote the relationship here with time as the dependent variable. Since the log function is the inverse of the exponential, his equation has t (time) equal to a (somewhat complicated) logarithmic function of p (population). Thus "logistic" is meant to convey the curve's "log-like" quality. For further discussion of this point, see Shulman [1997].

Exercises

27. Perform the integration indicated, and verify the equation for t. Why doesn't tM appear in the expression for t? (Recall that $M = \log e$.)

28. Graph t as a function of p.

29. Recall Yule's statement: "Verhulst now christens the curve a 'logistic.' He develops the principal properties, pointing out that the curve is symmetrical with respect to the point of inflection, and that the ordinate at the point of inflection is half the limiting ordinate." Verify that the curve is symmetrical with respect to the point of inflection, and that the ordinate at the point of inflection is half the limiting ordinate.

30. Express p as a function of t. Do you expect this curve to have the same properties? Graph p as a function of t.

2.6 Further History and Yule's Own Development

Yule continues his story.

But the work of Verhulst, as I have said, was forgotten. Only some four years ago, Professors Pearl and Reed of the Johns Hopkins University, Baltimore, working on interpolation formulae for population with especial reference to the United States, arrived independently at precisely the same result. After trying sundry purely empirical formulae, they point out that no such formula can be regarded as a general law of population growth, however good it may prove for practical purposes over a limited period. General considerations suggest something as to the form of the rational law. As there must be some limit to the population on the given area, the curve of growth must, sooner or later, turn over (i.e., in mathematical terms pass through a point of inflection) and gradually approach that limit. If we assume that the *absolute* rate of growth of the population (i.e., the numbers added to the population per unit of time, not the percentage rate of increase) is proportional to (1) the magnitude of the population existing at that instant, (2) "the still unutilized reserves of population-support existing" in the confined area, or in other words the differences between the existing and the limiting population, we arrive at precisely the form of law suggested by general considerations, and the formula is that given by Verhulst. Pearl and Reed's discovery was, however, quite independent, and their work on this subject seems to me of the highest importance and interest for the theory of population . . . .

I prefer to write Verhulst's formula for the law of growth in the form

$$y = \frac{L}{1 + e^{(\beta - t)/\alpha}}. \qquad \textbf{(1)}$$

where y is the population, t the time and L the limiting value of the population, which is only approached as t becomes indefinitely great. [Yule 1925, 4–5]

Exercise

31. Compare this with the equation that you obtained in **Exercise 29** above.

There are two other constants in addition to L, viz., α and β. Of these, α determines the horizontal scale of the curve—the greater α the more the curve spreads out—and I propose to call it the *standard interval*: β is the time from the zero of the time-scale to the point of inflection. Not to make the text of my Address too technical, I have relegated to Appendix II some discussion of the mathematics of the curve, which, following Verhulst, we may term a "logistic." Here it is only necessary to direct attention to some of its principal properties. If we choose the point of inflection as zero time, the standard interval as our unit of time, and the limiting population L as the unit of population, the formula (1) takes its simplest form

$$y' = \frac{1}{1 + e^{-\tau}}.$$

Fig. 1 [on the following page] shows the curve drawn from this formula. It starts rising very slowly and near the base line, gradually turns up more and more steeply till it reaches the point of inflection, and then gets flatter and flatter as it approaches the limit. It is symmetrical round the point of inflection, in the sense that if y' and y'' are ordinates of the curve equidistant from the point of inflection to left and right

$$y' = 1 - y''.$$

This is clearly a limitation on the generality of the curve, but only experience can tell us how far the symmetrical form is likely to be valid: both Verhulst and Pearl and Reed discuss more general forms. The proportional rate of increase at any instant of time, in the curve drawn as in fig. 1, is given by the complement of the ordinate, i.e., the intercept between the curve and the horizontal line representing the limiting population. It is obvious from the figure that at first, when the population is still very small, the proportional rate of increase only changes very slowly, so that the growth of the population can hardly be distinguished from growth in geometric progression; but as time goes on it falls more and more rapidly until the point of inflection is passed. It is important to note that in such a curve the proportional (or percentage) rate of increase of the population *falls continuously* from the start; if the percentage rate of increase of a population is steadily *rising* (mere disturbances excluded) it cannot be regarded as following a simple logistic cycle. It may be that such a population is passing from a cycle with a longer standard-interval to a cycle with a shorter standard-interval, e.g., when an agricultural country starts developing industries: or it may be that the population should be regarded as a mixture or association of two distinct populations following separate cycles.

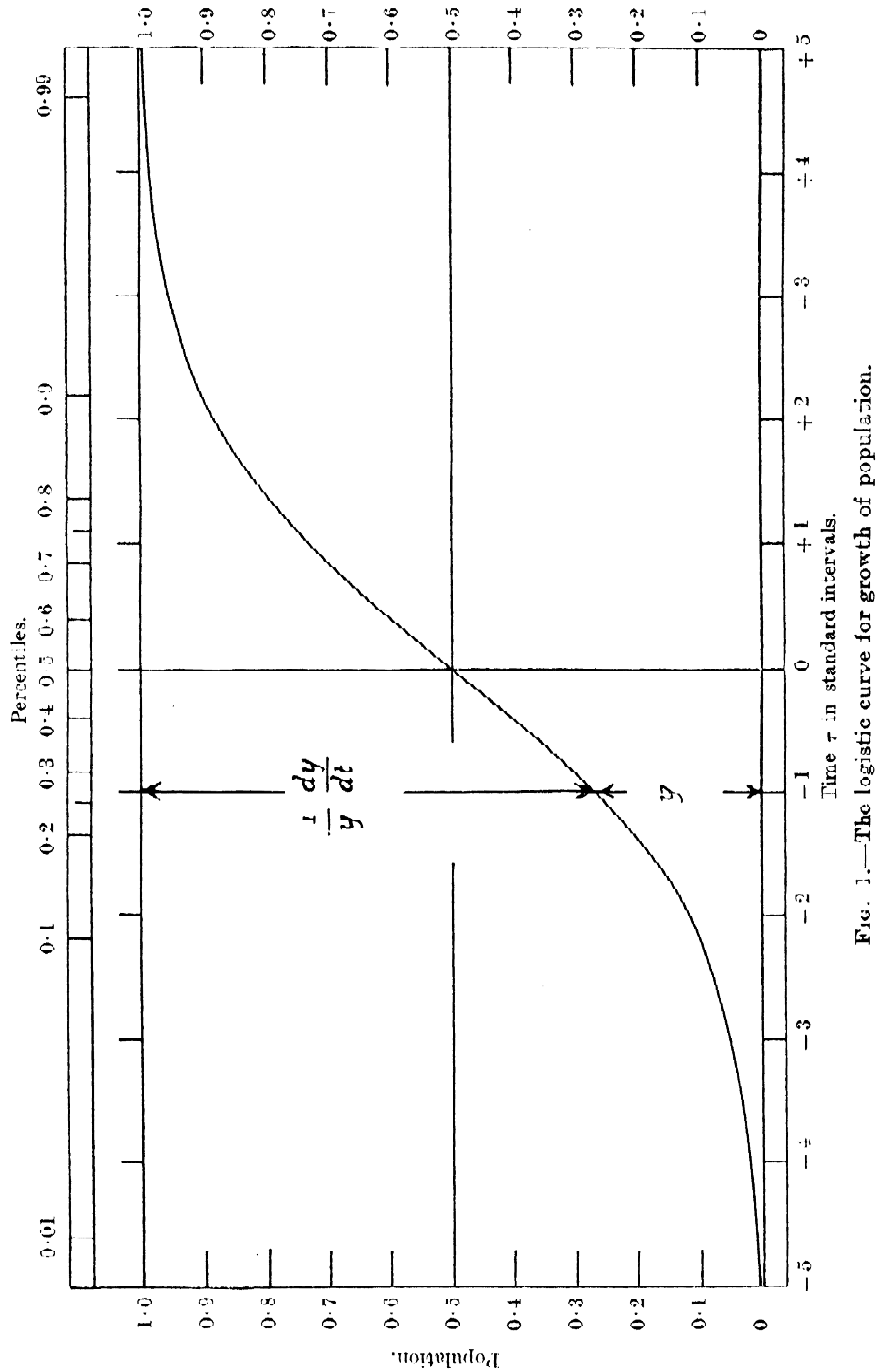

Figure 1 from Yule [1925, 6].

From the scale of percentiles at the top of fig. 1 it will be seen that the population stands at only 1 per cent of its limiting values at -4.6τ and reaches 99 percent of the limiting value at $+4.6\tau$. It therefore passes through the great bulk of the cycle in 9 or 10 standard intervals. The quartile and decile points lie at 1.1 and 2.2 respectively, very nearly indeed. The central 80 per cent of the range from 0 to L is therefore covered in roundly 4.4 standard intervals, and the central half in only 2.2 intervals.
[Yule 1925, 5–7]

2.7 Yule's Appendix

APPENDIX II. – – *Some notes on the mathematics of the logistic curve and methods of fitting*[2]. Let the differential equation be written: – –

$$(1/y)dy/dt = (1/\alpha)(1 - (y/L)) \tag{1}$$

In this form of the equation L is evidently the limiting population, since dy/dt is zero when $y = L$, and the constant α must be of the dimensions of a time. I shall term it the "standard interval," by analogy with the "standard deviation." The solution of the differential equation is

$$y = \frac{L}{1 + e^{(\beta - t)/\alpha}}.$$

where β is a constant of integration, and is evidently also a time. When t is infinite $y = L$: when $t = \beta$, $y = L/2$. But, differentiating (1) again,

$$d^2y/dt^2 = (1/\alpha)(1 - (2y/L)),$$

and hence $y = L/2$, $t = \beta$, gives the point of inflection. Further,

$$\begin{aligned} y_{\beta+h} &= L/(1 + e^{-h/a}) = L - L/(1 + e^{h/a}) \\ &= L - y_{\beta+h}. \end{aligned}$$

Hence the curve is symmetrical about the point of inflection.

The smaller y is compared with L the more nearly does the differential equation approach the simple form

$$(1/y)(dy/dt) = 1/\alpha.$$

But the solution of this is a logarithmic [we would say exponential] curve

$$y = Ae^{t/\alpha}.$$

That is to say, the early stages of the logistic are sensibly the same as a logarithmic [exponential] curve, or the curve of a geometric progression.

[2]For a more modern approach, see, e.g., Cavallini [1993].

We would then, in any case, expect the early stages of the growth of a population to be appreciably geometric; there does not seem to be any necessity for Verhulst's conception of an initial stage in which the growth is strictly geometric, passing abruptly into the logistic when the "normal population" is reached.

If we measure time with the standard interval as unit, denoting the time so measured by τ, take the point of inflection as the origin for time, and measure population with the limiting population L as unit, writing y' for y/L, we have

$$y' = \frac{1}{1+e^{-\tau}}. \qquad \textbf{(2)}$$

This is the simplest form of the equation to the logistic, and its differential equation is

$$(1/y')(dy'/dt) = 1 - y'.$$

Evidently we could draw such a logistic once and for all and fit the data for any actual population thereto by (1) replacing the actual populations by their ratios to the limiting population, (2) making the points of inflection coincide, (3) taking the standard time as the unit of our time-scale.

[Yule 1925, 46–47]

Exercise

32. Take three sets of data on various populations, "normalize" them as Yule suggests above, and then fit them to the normalized curve **(2)**.

3. Moral

The moral of this story[3] is:

The fundamental property of the logistic is that the instantaneous percentage rate of increase is a linear function of the population (equation (1)).

[Yule 1925, 48]

---

[3]But the story does not end here. The logistic model continues to yield new and exciting mathematics. For instance, the discrete-time version of the logistic population model can lead to chaotic dynamics. See, e.g., Schroeder [1991, 268ff].

4. Solutions to Selected Exercises

3.

Year	0	25	50	75	100	...	$25n$
Population	P_0	$2P_0$	$4P_0$	$8P_0$	$16P_0$	...	2^nP_0

Since $16P_0 = 112$, $P_0 = 7$ million.

4. In 225 years, the population doubles nine times; $7 \times 2^9 = 3{,}584$ million.

5. Let x be the number of millions supported. Then, since

$$\frac{7 \times 2^9}{x} = \frac{512}{10} = \frac{2^9}{10},$$

we have $x = 70$; so the answer is 70 million.

6.

Year	...	100	125	...	225
People supported	...	35	$35 + x$	...	$35 + 5x$

We have $35 + 5x = 70$, so $x = 7$. So the answer is 7 million every 25 years.

Alternatively, consider that we start by supporting $P_0 = 7$ million people. The number doubles in 25 years, so we have $14 = 7 + x$, or $x = 7$; and we get the same answer as before, 7 million every 25 years.

7. a)

Minute	0	1	2	3	...	n
Number of bacteria	2	4	8	16	...	2^{n+1}

We have $2^{n+1} = 2^{10}$, hence $n + 1 = 10$ and $n = 9$ min.

b)

Minute	0	1	2	3	...	n
Number of bacteria	2	4	6	8	...	$2(n+1)$

We have $2(n + 1) = 2^{10}$, hence $n + 1 = 2^9$ and $n = 2^9 - 1 = 511$ min.

9. a) The population grows by a factor of $\left(\frac{91{,}972}{3{,}929}\right)^{1/12} = 1.3005$ each decade, or by a factor of $(1.3005)^{1/10} = 1.0266$ each year.

b) $(91,972 - 3,929)/120 = 734$ thousand / year.

12. $dy/dx = ay(b - y)$.

18. $p = Ke^{mt}$.

30. $p = \dfrac{mbe^{mt/M}}{m - nb + nbe^{mt/M}}$.

References

Abers, Ernest, and Charles Kennel. 1977. *Matter in Motion: The Spirit and Evolution of Physics.* Boston, MA: Allyn and Bacon, Inc.

Cavallini, Fabio. 1993. Fitting a logistic curve to data. *College Mathematics Journal* 24: 247–253.

Kuhn, Thomas. 1970. *The Copernican Revolution.* Cambridge, MA: Harvard University Press.

Malthus, T.R. 1798. *An essay on the principle of population, etc.* London: Anon.

Pearl, Raymond. 1922. *The Biology of Death.* London, England: J.B. Lippincott.

________, and Lowell Reed. 1920. On the rate of growth of the population of the United States since 1790 and its mathematical representation. *Proceedings of the National Academy of Sciences* 6 (6) (15 June 1920): 275–288.

Schroeder, Manfred. 1991. *Fractals, Chaos and Power Laws: Minutes from an Infinite Paradise.* New York: W.H. Freeman.

Shulman, Bonnie. 1997. MATH-ALIVE!: Using Original Sources to Teach Mathematics in Social Context. *PRIMUS: Problems, Resources and Issues in Undergraduate Studies,* forthcoming.

Verhulst, Pierre-François. 1845. Recherches mathématiques sur la loi d'accroissement de la population. *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-Lettres de Bruxelles* 18: 1–38.

Yule, G. Udny. 1925. The growth of population and the factors which control it. *Journal of the Royal Statistical Society* 88, Part 1 (January 1925): 1–90.

About the Author

Bonnie Shulman received a Ph.D. in mathematical physics from the University of Colorado at Boulder in 1991. Since then she has been Assistant Professor of Mathematics at Bates College in Lewiston, ME. Her research interests include tracing the historical development of mathematical concepts, which she then incorporates in her teaching. She hopes that her students enjoy it as much as she does!