To Henry

From Sol

Henry Pollak has officially retired as a contributor to *Consortium*. He will turn 98 in December and although he is as sharp as ever, he would rather not have to deal with deadlines from pestering editors. So those editors and a few of us who have worked closely with Henry over the years thought that we would take this opportunity to say thank you.

For myself, I owe my career in mathematics education and COMAP's success to Henry. Our friendship goes back to ICME in Berkeley in 1980. COMAP was in its infancy as was I when it came to work in education. Henry of course was a giant. I needed to decide on whether to move from academic life and make COMAP more than an idea but a real place with a real staff and a real home for mathematical modeling. From that decision and through his years on the COMAP board, Henry's encouragement and support told me that I was doing something right. No matter what others said. No matter whether this grant or that got funded or not. If Henry respected our work then I knew it was worth doing.

I am not sure that I ever fully expressed to him what he meant to me personally and what his support meant to COMAP's existence and success. Thank you, Henry. You are a mensch!

In putting this piece together I asked Landy Godbold and Joe Malkevitch to chime in, as both have worked with Henry over the years. Landy shared a letter of support that he wrote for Henry's Dolciani award from MAA and Joe has supplied us with a Henry styled modeling problem as their ways of saying thanks.

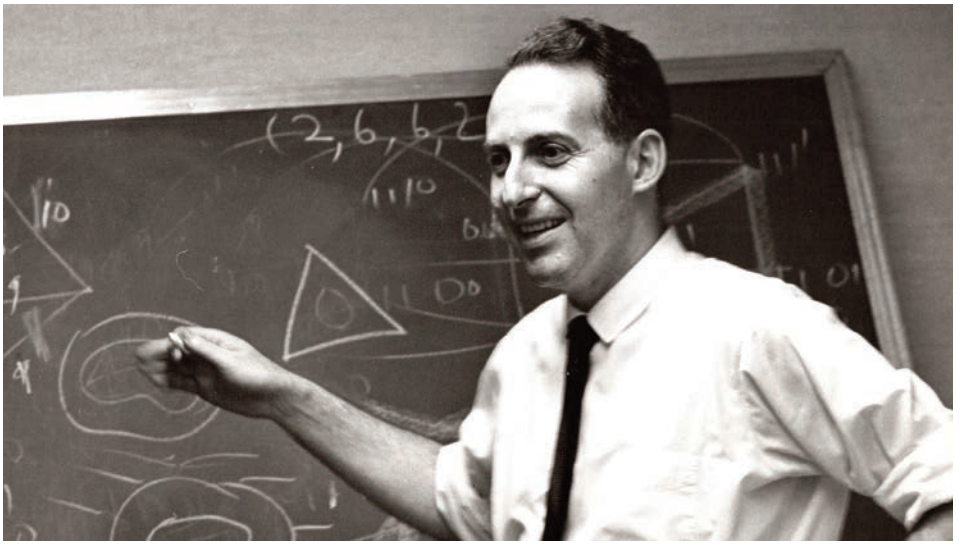


Photo Courtesy of Katherine Moldave

From Landy

Dr. Pollak's academic and professional credentials are extensive, reflecting numerous contributions to mathematics education in American schools for half a century. His vita provides ample evidence of his contributions to education in the large, so I'd like to use this letter to comment on his contributions on a smaller scale. I have had the good fortune to work with Henry on several projects over the past thirty years, and his attention to individuals has been at least as important as his formal mathematical presentations. That said, his talk at an MAA Section meeting in the 70s introduced me to modeling, which in turn changed the course of my career forever.

My first observation of Henry at work occurred soon after, at Georgia's Governor's Honors Program, a residential summer program for high school students interested and talented in mathematics. (His "fee" for visiting and giving lectures for the students was a basket of Georgia peaches.) Throughout his time on campus, Henry listened to students as they described for him their projects for the summer. His genuine interest in their work—questioning, prodding, probing, complimenting—had each student walking on air for a week.

That same kind of "leadership as peer" was his hallmark at the Woodrow Wilson National Fellowship Foundation's Summer Institutes, held on the campus of Princeton University from 1984 through 1992, for a new group of 50 select teachers from around the country each year. A major goal of the program was to develop teachers who would "go home" and teach other teachers. I had the good fortune of attending the first one and then of serving on the staff for the final five years. Again, Henry gave numerous formal talks on a wide range of mathematical topics, but his individual counsel shaped summer projects, mathematical self-confidence, and career paths. He delighted in *learning* from participants—new ways to solve "old" problems or pedagogical strategies. He enjoyed the role of student; he was not there to be the "fount of all knowledge."

His interest in the participants extended to discussions regarding individuals' "real" lives, such as upcoming decisions concerning grad schools and the like.

That same focus on the individual—as a person, a teacher, a mathematician, and a modeler—was in full force in his capacity of co-PI on the NSF curriculum project ARISE for COMAP in the 1990s. In short, Henry put everyone at ease—always teaching, but not very often in an overt manner. He is the personification of the life-long learner, and that, together with his genuine interest in those with whom he comes in contact, makes him the teachers' teacher. I know that my career would have looked vastly different had I not crossed paths with him, and I am certain that thousands of others would echo that sentiment, even without all his publications and lectures.

From Joe

Henry O. Pollak has been a vocal advocate for the teaching of mathematical modeling as one of the essential skills that should be part of everyone's pre-college education. Being able to *use* the mathematics one learns in school in everyday life often got lost in the details of learning the techniques associated with arithmetic, algebra and geometry. If adding 3/4 and 7/10 is an important skill perhaps it is also an important skill to be able to give an example of a situation in your life where you care about what the answer would be! Henry's career path molded his views. His training involved getting a doctorate degree in 1951 in mathematics at Harvard University under the direction of Lars Ahlfors (who shared winning mathematics' most prestigious award of the time, the Fields Medal, in 1936, the first year the prize was awarded). While often mathematicians who finish a doctorate enter academia to teach and continue doing research, Henry joined Bell Labs, the research arm of the Bell Telephone System in 1951. Eventually he rose to become director of the Mathematics and Statistics Research Center of Bell Labs. On leaving Bell Labs he joined the mathematics education faculty at Teachers College Columbia, where he championed the teaching of mathematical modeling for future teachers, not only those who were specializing in teaching mathematics. A characteristic of Henry's approach to getting insight into applied mathematics and mathematical modeling was generating situations that a typical person might be faced with and gave them to students to work on and analyze either as an individual or in a group learning setting. In the spirit of what Henry did so remarkably well, consider the problem below in the style of a "Henry Problem."

Sharing the Rent

Andrew (A), Bob (B), Cheryl (C) and David (D) are friends from attending the same high school and wish to get a running start at college X in town Z which they all will attend in the Fall. The friends plan on participating in a summer school program. They will rent a house for the coming summer, even though starting in the Fall they must be living in dormitories at college X. Online they have seen a house advertised (including pictures which show all the rooms of the house they would share) which they have made an arrangement to rent for $3200 for the whole summer. However, they are faced with the issue of how to divide the rent among themselves and which room each of them should occupy.

The current teacher (Dr. Wise) of their AP Mathematics class, which all of them are enrolled in, has agreed to work out which student would get assigned to each of the rooms in the house and what part of the rent they would each pay.

8 CONSORTIUM Henry's Notes

POSTCARD COLLECTING

HENRY POLLAK

One of my minor vices—some would call it an addiction—is that I am a collector. One of my collections is of lithographed postcards published by one of a number of firms associated with the family named Schwidernoch. They started in Vienna in 1891, and the business reached a peak in the period 1897–1899. The postcards were of the "greetings from" variety and depicted places primarily in the Austro-Hungarian Empire. My guess is that they produced in the neighborhood of 6,000 different lithographs. They are beautifully done, and much in demand. Many, but not all, of the cards are numbered, apparently in sequence of arriving first-time orders. Reorders generally kept the old card number.

I should like to estimate how many different cards there were of all places in a particular province. The example I shall use is what was a territory formerly under Turkish administration, and at that time occupied by Austria, namely Bosnia-Herzegovina (BH). How many different cards depicting communities in BH did the Schwidernochs produce?

Initially, all I knew is how many I had. This number has been growing slowly, since I like them and look for them, and in the middle of 2005 had reached 18. At that point, a book was published that showed 7 BH lithographs in the author's collection, and I found that I had 3 of those 7. For the first time, I could imagine a solvable mathematical problem—very far from reality in its assumptions, but at least a first step. Let us assume that the Schwidernochs produced n different BH lithographs, each equally likely to be in each of the two collections (which is highly dubious in reality), and that the two collections had been gathered independently (true up to that point, but not if we ever made a trade.) Thus we know that there must be at least 22 different cards (3 in both collections, 4 only in his, and 15 only in mine.) But under the assumption of equally likely selection from n cards, if n were 22, it is very unlikely that we would have as few as 3 in common. On the other hand, if n were very large, it is very unlikely that we would have as many as 3 in common!

Somewhere between 22 and infinity, there is a value of n that would maximize the probability that collections of 18 and 7 cards would have exactly 3 in common. What I am looking for is this *maximum likelihood estimate* of n.

At this initial stage, the problem, but perhaps not the context, may well look familiar to you. It seems to be exactly the capture-recapture method of estimating N, the number of fish in a lake. Imagine going to the lake on day 1, capturing 18 fish, and tagging and releasing them unharmed. On day 2, capture 7 fish, observe that 3 of them are tagged, and ask for the maximum likelihood estimate of N. This is a familiar problem; let us remind ourselves of its solution. But I will stick to the postcard terminology.

Suppose that there are n equally likely different cards altogether. Suppose that person A has a cards, person B has b cards, and that they have c in common. For convenience, I think of a as at least as large as b. Then the number of ways that this can happen can be computed as follows: Given that B has b cards out of the n that exist, B picks c cards out of the a that A has, and picks his remaining $(b-c)$ cards out of the remaining $(n-a)$ cards. The probability $p(a, b, c; n)$ that A has a cards, B has b cards, A and B have c cards in common, given that there exist n different cards in total, is

$$p(a,b,c;n) = \frac{\binom{a}{c}\binom{n-a}{b-c}}{\binom{n}{b}}$$

Mathematical topic from Consortium 90

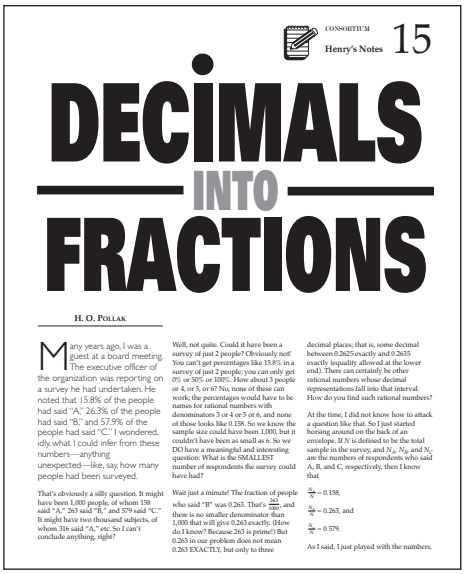
Data about the house:

a. Ground floor: Living room; kitchen, full bath with tub and shower.

b. Second floor: Bathroom with toilet and shower; Room A: 8×10, single bed; window; Room B: 12×14; two twin beds, with a small toilet, no window; Room C: 11×11 double bed, no window.

Dr. Wise has requested that each student submit their family income from the prior tax year. The students would also submit which friends, if assigned the "double" room, they would be willing to have as a roommate or if sharing a room is unacceptable.

Comment: it is known that at least one of the possible 6 pairs that might share a room are willing to share a room, but all of the students would prefer to have their own room.) Initially, Dr. Wise had thought to ask the students to rank all three rooms without ties from best to worst choice (ordinal ballot) but after some thought had decided to use a type of ballot that perhaps takes more effort to prepare but might provide "better"


CONSORTIUM Henry's Notes 15

DECIMALS INTO FRACTIONS

H. O. Pollak

Many years ago, I was a guest at a board meeting. The executive officer of the organization was reporting on a survey he had undertaken. He noted that 15.8% of the people had said "A," 26.3% of the people had said "B," and 57.9% of the people had said "C." I wondered, idly, what I could infer from these numbers—anything unexpected—like, say, how many people had been surveyed.

That's obviously a silly question. It might have been 1,000 people, of whom 158 said "A," 263 said "B," and 579 said "C." It might have two thousand subjects, of whom 316 said "A," etc. So I can't conclude anything, right?

Well, not quite. Could it have been a survey of just 2 people? Obviously not! You can't get percentages like 15.8% in a survey of just 2 people; you can only get 0% or 50% or 100%. How about 3 people or 4, or 5, or 6? No, none of these can work; the percentages would have to be names for rational numbers with denominators 3 or 4 or 5 or 6, and none of those looks like 0.158. So we know the sample size could have been 1,000, but it couldn't have been as small as 6. So we DO have a meaningful and interesting question: What is the SMALLEST number of respondents the survey could have had?

Wait just a minute! The fraction of people who said "B" was 0.263. That's $\frac{263}{1000}$, and there is no smaller denominator than 1,000 that will give 0.263 exactly. (How do I know? Because 263 is prime!) But 0.263 in our problem does not mean 0.263 EXACTLY, but only to three decimal places; that is, some decimal between 0.2625 exactly and 0.2635 exactly (equality allowed at the lower end). There can certainly be other rational numbers whose decimal representations fall into that interval. How do you find such rational numbers?

At the time, I did not know how to attack a question like that. So I just started horsing around on the back of an envelope. If N is defined to be the total sample in the survey, and N_A, N_B, and N_C are the numbers of respondents who said A, B, and C, respectively, then I know that

$\frac{N_A}{N} = 0.158$,

$\frac{N_B}{N} = 0.263$, and

$\frac{N_C}{N} = 0.579$.

As I said, I just played with the numbers,

Mathematical topic from Consortium 92

information. For this other type of ballot (range ballot) each voter would assign each room some number of points with 0 low, 100 points high, and two or more rooms could be assigned the same number of points.

Problem:
Design a system for Dr. Wise to assign the students to the room based on

a. Family income and the ordinal ballots provided.

b. Family income and the range ballots provided.

Comments:

a. What "optimization" function do you think Dr. Wise should use in deciding the best assignment?

b. What "fairness" axioms (rules) might the system Dr. Wise devises be required to obey.

For your benefit some of the terms that are sometimes associated with problems of this kind are:

a. Envy free; ordinal ballots; cardinal ballot; approval ballot.

b. One can add "utilities" or value points of different kinds.

From Paul

As the latest pestering editor of *Consortium*, and so entitled to a final thought, I want to personally acknowledge the tremendous value of Henry's contributions to not only COMAP but to mathematics education more broadly. As highlighted above, students and teachers are far better for the attention he devoted to the intersection of mathematics, the real world, teaching, learning, and the people around him. My career trajectory, and even more so, my approach to both mathematics and the teaching of mathematics owe a great deal to my first encounter with Henry when I was one of the participants in the 1988 Woodrow Wilson Summer Institute.

A high school teacher of only three years' experience, I was sitting between two veteran teachers during a computer lab in which we had solved some problem with a minimum spanning tree. Given the advent of computers, my inclination was to explore a computational space in ways I never would by hand. So I mused aloud about the relation of the minimum tree to the distribution of all spanning trees. One of the veteran teachers said, somewhat scathingly, "Why would you want to do that?!" Before I could compose a reply or decide whether to begin coding, Henry leaned over my shoulder and said, "That's an interesting idea, let's see!"

I don't recall exactly where those computations led, but the object lesson of encouraging exploration, even after a solution was in hand, stayed with me. Ever since that nudge from Henry, I have been on the lookout for students who, with their perhaps off the beaten track questions, might just need encouragement to take their thinking a bit further.

Thank you Henry for setting in motion countless explorations of mine, my students, and countless others.

From COMAP

Henry, we will miss your column, but you will always be at the heart and soul of what COMAP is and does. So, from all of us and your readers – thank you Henry.

CONSORTIUM Henry's Notes 3

CYCLING OF EXPENSES

H. O. Pollak

One of the enjoyable features of mathematical research at the old Bell laboratories was the great variety of problems that would "walk in the door." They came not just from the rest of the Labs, but from all over the Bell System. Not all were interesting, but the variety was fine for any mathematician who liked more than just one area of mathematics. Any narrow picture of "applicable" mathematics was quickly dispelled.

Sometime in 1958, my long-term colleague Ed Gilbert and I were visited by Mr. A. M. Marca of the Western Electric Company, which was at that time the manufacturing arm of the Bell System. He wanted to know if the method they used for solving a particular accounting problem could be simplified, and if he could, in fact, be assured that it would always work.

Here was the situation. A manufacturing organization has a large number of *producing* departments, and it is easy to assign the costs of running such departments to the products they manufacture. There are also *service* departments, such as stock rooms, food service, janitorial service, and secretarial service. The costs of running these have to be allocated somehow to the products of the manufacturing departments. At first glance, this does not appear to be much of a problem: Manufacturing department A has a certain number of employees, and this determines its share of the cost of running the restaurant.

But the problem is more difficult: The secretaries also eat, and the management of the restaurant gets its letters typed. In other words, the service departments service not only the producing departments, but they also service each other! What is a fair way of taking this into account?

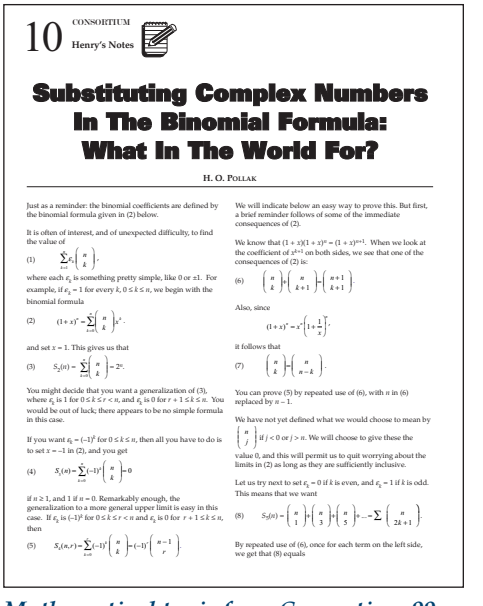
At Western Electric at that time, this was done by an iterative procedure called *cycling of expenses*. A successive iteration method for approximating and then allocating the cost of inter-service-department service was used until the cost had settled to the nearest dollar. Experience showed that the method always worked, but they would have preferred to be sure of that, and they wanted to know if there was a simpler way to go about it.

MATHEMATICAL FORMULATION

A possible approach to computing the actual cost of a service department is to add to the raw, direct cost of running the service department all the service it receives from other service departments. Of course this makes it look as if the service departments cost more than they actually do, and one might have to admit that this might even be considered dishonest. We shall see, in the discussion immediately preceding Exercise 2, that it actually isn't!

Let the *service* departments be called D_j, $i = 1, 2, \ldots, n$, and let us assume we know each of the following quantities:

Mathematical topic from Consortium 93


10 CONSORTIUM Henry's Notes

Substituting Complex Numbers In The Binomial Formula: What In The World For?

H. O. Pollak

Just as a reminder: the binomial coefficients are defined by the binomial formula given in (2) below.

It is often of interest, and of unexpected difficulty, to find the value of

$$(1) \qquad \sum_{k=0}^{n} \varepsilon_k \binom{n}{k},$$

where each ε_k is something pretty simple, like 0 or ±1. For example, if $\varepsilon_k = 1$ for every k, $0 \le k \le n$, we begin with the binomial formula

$$(2) \qquad (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

and set $x = 1$. This gives us that

$$(3) \qquad S_0(n) = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

You might decide that you want a generalization of (3), where ε_k is 1 for $0 \le k \le r < n$, and ε_k is 0 for $r+1 \le k \le n$. You would be out of luck; there appears to be no simple formula in this case.

If you want $\varepsilon_k = (-1)^k$ for $0 \le k \le n$, then all you have to do is to set $x = -1$ in (2), and you get

$$(4) \qquad S_1(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

if $n \ge 1$, and 1 if $n = 0$. Remarkably enough, the generalization to a more general upper limit is easy in this case. If ε_k is $(-1)^k$ for $0 \le k \le r < n$ and ε_k is 0 for $r+1 \le k \le n$, then

$$(5) \qquad S_1(n,r) = \sum_{k=0}^{r} (-1)^k \binom{n}{k} = (-1)^r \binom{n-1}{r}.$$

We will indicate below an easy way to prove this. But first, a brief reminder follows of some of the immediate consequences of (2).

We know that $(1+x)(1+x)^n = (1+x)^{n+1}$. When we look at the coefficient of x^{k+1} on both sides, we see that one of the consequences of (2) is:

$$(6) \qquad \binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

Also, since

$$(1+x)^n = x^n \left(1 + \frac{1}{x}\right)^n,$$

it follows that

$$(7) \qquad \binom{n}{k} = \binom{n}{n-k}.$$

You can prove (5) by repeated use of (6), with n in (6) replaced by $n-1$.

We have not yet defined what we would choose to mean by $\binom{n}{j}$ if $j < 0$ or $j > n$. We will choose to give these the value 0, and this will permit us to quit worrying about the limits in (2) as long as they are sufficiently inclusive.

Let us try next to set $\varepsilon_k = 0$ if k is even, and $\varepsilon_k = 1$ if k is odd. This means that we want

$$(8) \qquad S_2(n) = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = \sum \binom{n}{2k+1}.$$

By repeated use of (6), once for each term on the left side, we get that (8) equals

Mathematical topic from Consortium 99