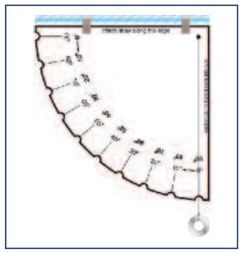
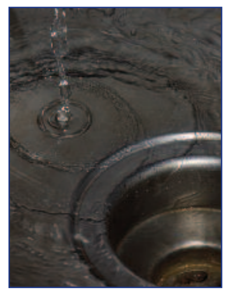
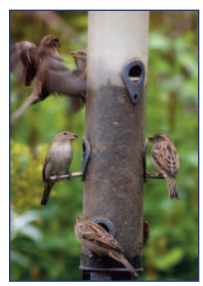
TEACHERS COLLEGE
COLUMBIA UNIVERSITY

Mathematical Modeling Handbook







COMAP

TEACHERS COLLEGE
COLUMBIA UNIVERSITY

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Mathematical Modeling Modules

PREFACE

For most teachers, mathematical modeling represents a new way of "doing" mathematics that makes the addition of modeling activities into instruction seem daunting. This is especially true since modeling, when done properly, requires significant time and effort. In turn, some may be reluctant to include modeling activities into classroom time. It is essential to keep in mind that modeling is one of the eight Standards for Mathematical Practice given in the *Common Core State Standards for Mathematics* (CCSSM) for all grades and is a required conceptual category in high school. Because of this, *modeling cannot be set aside or employed only when spare time arises*. Class time that previously may have been spent using more traditional teaching methods should be converted to time spent on modeling. The integrated nature of mathematical modeling, and in turn the number of curricular standards covered when working through a modeling activity, make modeling activities a very efficient use of class time.

The *Teachers College Mathematical Modeling Handbook* is intended to support the implementation of the CCSSM in the high school mathematical modeling conceptual category. The CCSSM document provides a brief description of mathematical modeling accompanied by 22 star symbols (*) designating modeling standards and standard clusters. The CCSSM approach is to interpret modeling not as a collection of isolated topics but in relation to other standards.

The goal of this *Handbook* is to aid teachers in executing the CCSSM approach by helping students to develop a *mathematical disposition*, that is, to encourage recognition of mathematical opportunities in everyday events. The *Handbook* provides modules and guides for 26 topics together with references to specific CCSSM modeling standards for which the topics may be appropriate. It should be noted that only those standards that have been marked specifically as modeling standards are listed within the modules, however many more standards not marked specifically for modeling are covered.

The *Handbook* begins with an introductory essay by Henry O. Pollak entitled "What is Mathematical Modeling?" Pollak joined the Teachers College faculty in 1987 where he has continued his involvement in modeling and its teaching, emanating from his three decades of work at Bell Laboratories. Pollak has contributed to other COMAP projects and publications including *Mathematics: Modeling Our World* (2000) and "Henry's Notes" in the newsletter *Consortium*.

Each module is presented in the same format for ease of use. Each contains four sections: (1) Teacher's Guide – Getting Started, (2) student pages (comprising the student activities), (3) Teacher's Guide – Possible Solutions, and (4) Teacher's Guide – Extending the Model.

The first section of each module, "Teacher's Guide – Getting Started", is for teachers only. It contains information similar to that in a typical lesson plan: it is meant to give an overview of the module including motivation, the amount of time necessary, what materials and prerequisite knowledge are required, and a general outline describing the student activities in "Worksheet 1" (the first day's activities) and "Worksheet 2" (the second day's activities). At the end of this section, the CCSSM modeling standards the module is intended to cover are listed.

The next section of each module consists of the worksheet pages for students. These pages should be photocopied and distributed for student use. It is the teacher's choice how these lessons should be implemented, but the modules were written with the intention of being a combination of classroom discussion, group, and individual work. The first page of this section is an "artifact page" which lays the foundation for the scenario to be modeled. Occasionally, tables of information, helpful pictures, or tools to be used in the model are included – the so-called "artifacts". The artifact page concludes with the "leading question" that is the main idea to be addressed and is meant to drive the modeling activities.

The first day's lesson continues after the artifact page and consists of two or three pages. Questions are presented in such a way that students are expected to develop a model to begin to answer the leading question presented on the artifact page. By the end of the first day in most lessons, students craft their model either with mathematics or, sometimes, with actual, physical constructions.

The second day's lesson immediately follows the first day's lesson. It often begins with a "recap" of what happened previously. Definitions sometimes are listed to help drive the model in a mathematical direction in order to focus student attention on specific mathematical ideas. Students continue to work with and refine their model in an effort to answer the leading question more completely or accurately. Sometimes, the lesson proceeds beyond the idea originally posed to help students apply their model to different or more complex scenarios.

Throughout the student pages, there are bracketed notes intended to help guide students through more tdifficult problems. These are meant to be used if there is trouble moving on from the question and can help you, the teacher, guide the lesson in the direction necessary for completing a model.

The "Teacher's Guide – Possible Solutions" section follows the student pages. Possible answers to the questions posed on both days' student worksheets are given according to the numbered questions. This is intended to give teachers a general guide for how the lesson might progress, but is not meant to be a rigid structure by which classes must abide. Mathematical modeling often can be perceived within several disciplines: students' work should be based on mathematical validity and not on the ability to adhere to the strict mathematical idea presented in the modules.

Each module concludes with a section entitled "Teacher's Guide – Extending the Model" contributed by Pollak. Typically, three kinds of materials for interested and advanced students may be found there: possible extensions of the model developed in the module, other applications of the mathematics of the module, and mathematical extensions of the mathematics within the module. Models are not restricted to one idea and thus have many different uses. "Extending the Model" shows how this is possible.

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Heather Gould, Chair
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Reference – The CCSSM is referenced throughout this *Handbook*, but we will refrain from listing it within each module and only give it here.

National Governors Association Center for Best Practices, Council of Chief State School Officers. (2010). *Common core state standards for mathematics* (CCSSM). Washington, D.C.: National Governors Association Center for Best Practices, Council of Chief State School Officers.

INTRODUCTION
What Is Mathematical Modeling?

Henry O. Pollak
Teachers College, Columbia University

Mathematical Modeling in a Nutshell

Mathematicians are in the habit of dividing the universe into two parts: mathematics, and everything else, that is, the rest of the world, sometimes called "the real world". People often tend to see the two as independent from one another – nothing could be further from the truth. When you use mathematics to understand a situation in the real world, and then perhaps use it to take action or even to predict the future, both the real-world situation and the ensuing mathematics are taken seriously. The situations and the questions associated with them may be any size from huge to little. The big ones may lead to lifetime careers for those who study them deeply and special curricula or whole university departments may be set up to prepare people for such careers. Electromagnetic theory, medical imaging, and cryptography are some such examples. At the other end of the scale, there are small situations and corresponding questions, although they may be of great importance to the individuals involved: planning a trip, scheduling the preparation of Thanksgiving dinner, hiring a new assistant, or bidding in an auction. Problems of intelligent citizenship vary greatly in complexity: deciding whether to vote sincerely in the first round of an election, or to vote so as to try to remove the most dangerous threat to your actual favorite candidate; planning the one-way traffic patterns for your downtown; thinking seriously, when the school system argues about testing athletes for steroids, whether you prefer a test that catches almost all the users at the price of designating some non-users as (false) positives, or a test in which almost everybody it catches *is* a user, but misses some of the actual users.

Whether the problem is huge or little, the process of "interaction" between the mathematics and the real world is the same: the real situation usually has so many facets that you can't take everything into account, so you decide which aspects are most important and keep those. At this point, you have an idealized version of the real-world situation, which you can then translate into mathematical terms. What do you have now? A *mathematical model* of the idealized question. You then apply your mathematical instincts and knowledge to the model, and gain interesting insights, examples, approximations, theorems, and algorithms. You translate all this back into the real-world situation, and you hope to have a theory for the idealized question. But you have to check back: are the results practical, the answers reasonable, the consequences acceptable? If so, great! If not, take another look at the choices you made at the beginning, and try again. This entire process is what is called *mathematical modeling*.

You may be wondering how mathematical modeling differs from what you already teach, particularly, "problem solving". Problem solving may not refer to the outside world at all. Even when it does, problem solving usually begins with the idealized real-world situation in mathematical terms, and ends with a mathematical result. Mathematical modeling, on the other hand, begins in the "unedited" real world, requires problem *formulating* before problem *solving*, and once the problem is solved, moves back into the real world where the results are considered in their original context. Additionally, it would take us too far afield to discuss *whimsical* problems, where mythical kingdoms and incredible professions and procedures may become the setting of some lovely mathematics. They make no pretense of being problems motivated by the real world.

Mathematical Modeling and Education

Now that we have an idea about what mathematical modeling is in the real world, what do we do about it in mathematics education? One hundred years ago, the big areas – classical physics, astronomy, cartography, and surveying, for instance – were taught in university mathematics departments, perhaps called departments of mathematics and astronomy. Nowadays, in the United States at least, these are taught in science or engineering departments. These branches of science are big and they are very old. What about areas that have become major appliers of mathematics during the last century? Information theory and cryptography may be included in the curricula of electrical engineering, inventory control, programming (as in "linear"), scheduling and queuing in operations research, and fair division and voting in political science. These topics are such exciting new areas of application, often of discrete mathematics, that they frequently have a home in mathematics, as well. Who is going to "own" them in the long run is undecided.

What do we, as mathematicians and mathematics educators, conclude? Many scientific disciplines use mathematics in their development and practice, and when they are faithful to the science they do indeed check which aspects of the situation they have kept and which they have chosen to ignore. Engineers and scientists, be they physical, social, or biological, have not expected mathematics to teach the modeling point of view for them within a scientific framework, although preparing for this kind of reasoning is part of mathematics. Since the scientists will do mathematical modeling anyway, can we just leave mathematical modeling to them? *Absolutely not*. Why not? Mathematics education is at the very least responsible for teaching how to use mathematics in everyday life and in intelligent citizenship, and let's not forget it. Actually, any separation of science from everyday life is a delusion. Both everyday life and intelligent citizenship often also involve scientific issues. So what really matters in mathematics education is learning and practicing the mathematical modeling process. The particular field of application, whether it is everyday life or being a good citizen or understanding some piece of science, is less important than the experience with this thinking process.

Mathematical Modeling in School

Let us now look at mathematical modeling as an essential component of school mathematics. How successfully have we done this in the past? What are the recollections, and the attitudes, of our graduates? People often say that the mathematics they learned in school and the mathematics that they use in their lives are very different and have little if anything to do with each other. Here's an example: the textbook or the teacher may have asked how long it takes to drive 20 miles at 40 miles per hour, and accepted the answer of 30 minutes. But how does all this come up in everyday life? When you live 20 miles from the airport, the speed limit is 40 mph, and your cousin is due at 6:00 pm, does that mean you leave at 5:30 pm? Your actual thinking may be quite different. This is rush hour. There are those intersections at which you don't have the right of way. How long will it take to find a place to park? If you take the back way, the average drive may take longer, but there is much less variability in the total drive time. And don't forget that the arrival time the airline's website gives you is the time the plane is expected to touch down on the runway, not when it will start discharging passengers at the gate. And so forth. Contrasting these two thought processes, there is no wonder that students don't see the connection between mathematics and real life.

We said at the beginning that in mathematical modeling, both the real-world situation and the mathematics are "taken seriously". What does that mean? It means that the words and images from outside of mathematics are not just idle decorations. It means that the *size* of any numbers involved is realistic, that the *precision* of the numbers is realistic, that the question asked is one that you would ask in the real world. It means that you have considered what aspects of the real-world situation you intend to keep and what aspects you will ignore.

A mathematical model, as we have seen, begins with a situation in the real world which we wish to understand. The particular branch of mathematics that you will end up using may not be known when you start. But then how do you know when and where in the curriculum to discuss a certain modeling problem? If you put it in a section on plane geometry, then students will look for a plane geometric model! Is that what you want? An answer to this difficulty, which is quite real, is that, as in all of mathematics, the learning and the pedagogy are spiral and you return to a major idea many times. In the student's first experiences with modeling, the particular mathematics to be used will be quite obvious, and that's fine. Later on, the student may have to consider some alternatives ("Should I try plane geometry, or analytic geometry, or vectors?"), but may very well need help in finding what these alternatives might be, and how to think about the consequences of picking any particular one. At an advanced level, such hints will, we hope, become less necessary.

The Variety of Modules

We have seen that modeling arises in many major disciplines within science, engineering, and even social sciences. As such, it will be at the heart of courses in many disciplines, and at the heart of many varied careers.

Mathematical modeling is also an important aspect of everyday life, where everyone will be better off if they become comfortable with it. It enters many facets of intelligent citizenship. Which kinds of situations do you want to emphasize in school? It is tempting to use modeling as an opportunity to get students thinking about the big issues of our time: world peace, health care, the economy, or the environment. The main point is to develop a favorable disposition and comfort with mathematical modeling, and big issues don't often fit into modules with two lessons.

So, tempting as it may be, the contents of this *Handbook* do not attack any of the major problems of the world. There are *some* modules that can be considered as giving a foretaste of a whole discipline. *A Model Solar System* points towards Newton, Kepler, and the laws of motion and astronomy. Periodic phenomena also appear in both natural and man-made systems, as can be seen in *Sunrise, Sunset*. *A Bit of Information* gives a taste of the very beginning of information theory, and *State Apportionment* starts some thinking about that particular fair division problem. An introduction to the modeling of epidemics can be associated with *Viral Marketing*.

Both continuous and discrete mathematics are important for modeling. *Bending Steel* and *Water Down the Drain* are examples of continuous problems. An important aspect of a modeling disposition is the ability to make "back-of-the-envelope" estimates that give insight into phenomena that are sometimes surprising. Both *Bending Steel* and the extension to *Water Down the Drain* partake of this aspect of modeling. On the other hand, *A Tour of Jaffa* is discrete, and *Sunken Treasure* has discrete, continuous, and even experimental aspects. And it is sometimes surprising that functions with piecewise definitions occur in the real world as often as they do. Such a problem involves both continuous and discrete thinking. *For the Birds* gives an unexpected example.

Quite naturally, the modules involve a wide variety of high school mathematical topics. Looking for a function with particular properties is at the heart of *A Bit of Information* (logarithms) and of *Rating Systems* (a logistic curve). Another method of looking for a function is to fit a curve to data, which is part of *A Model Solar System*. It is also part of the mathematical modeling process to progress through various areas of mathematics as you become more adept at a particular modeling situation. Thus geometry, algebra, and trigonometry are all part of the development of *Narrow Corridor*. *Sunken Treasure*, besides using a variety of forms of mathematical reasoning, even suggests using physics in order to do mathematics!

A number of other important mathematical ideas arise in the course of this collection of modeling problems. For example, in connection with several modules involving probability and statistics, the notion of optimal stopping occurs more than once. It is the central idea in *Picking a Painting* and has an important role in *The Whe to Play*. The Intermediate Value Theorem has a crucial role in *Unstable Table*, a delightful everyday-life application of mathematics towards having a comfortable meal in a restaurant. The logistic curve shows up in *Rating Systems* and Voronoi diagrams in *Gauging Rainfall*. Simple everyday-life situations are found in *For the Birds*, *Estimating Temperatures*, and *Changing It Up*. We do have one whimsical problem, *Flipping for a Grade*.

A fundamental aspect of mathematical modeling, as is emphasized many times in the *Common Core State Standards for Mathematics*, in the literature on modeling, and in the present work, is the fact that every model downplays certain aspects of reality, which in turn means that the mathematical results eventually have to be checked against reality. This may lead to successive deepening of the models, which shows up particularly in *Narrow Corridor*, *A Tour of Jaffa*, and *Unstable Table*. This may be viewed as a new facet of Polya's dictum, "look for a simpler problem".

It is time to bring this introductory essay to a close. We propose two codas, one for mathematicians and one for mathematics educators.

Coda for Mathematicians

We have discussed a number of examples to show the variety of experiences which this collection is intended to encompass. They illustrate situations from everyday life, from citizenship, and from major quantitative disciplines, situations chosen because they lend themselves to brief introductory experiences in mathematical modeling. Don't get the impression that all of this is an unnatural demand on mathematics education. Far from it, it strengthens the affinity between pure mathematics and its applications. The heart of mathematical modeling, as we have seen, is problem formulating before problem solving. So often in mathematics, we say "prove the following theorem" or "solve the following problem". When we start at this point, we are ignoring the fact that finding the theorem or the right problem was a large part of the battle. By emphasizing problem finding, mathematical modeling brings back to mathematics education this aspect of our subject, and greatly reinforces the unity of the total mathematical experience.

Coda for Mathematics Educators

Probably 40 years ago, I was an invited guest at a national summer conference whose purpose was to grade the AP Examinations in Calculus. When I arrived, I found myself in the middle of a debate occasioned by the need to evaluate a particular student's solution of a problem. The problem was to find the volume of a particular solid which was inside a unit three-dimensional cube. The student had set up the relevant integrals correctly, but had made a computational error at the end and came up with an answer in the millions. (He multiplied instead of dividing by some power of 10.) The two sides of the debate had very different ideas about how to allocate the ten possible points. Side 1 argued, "He set everything up correctly, he knew what he was doing, he made a silly numerical error, let's *take off* a point." Side 2 argued, "He must have been sound asleep! How can a solid inside a unit cube have a volume in the millions?! It shows no judgment at all. Let's *give* him a point."

My recollection is that Side 1 won the argument, by a large margin. But now suppose the problem had been set in a mathematical modeling context. Then it would no longer be an argument just from the traditional mathematics point of view. In a mathematical modeling situation, pure mathematics loses some of its sovereignty. The quality of a result is judged not only by the correctness of the mathematics done within the idealized mathematical situation, but also by the success of the confrontation with reality at the end. If the result doesn't make sense in terms of the original situation in the real world, it's not an acceptable solution.

How would you vote?

A NOTE ON TEACHER EDUCATION AND PROFESSIONAL DEVELOPMENT

While this *Handbook* was written with the goal of providing CCSSM-aligned, "ready-made" worksheets for high school teachers to distribute for student use, it also can be adapted easily for use in undergraduate teacher education programs, pre- and in-service training programs, and professional development. The editors would like to recommend some uses of the *Handbook* for those working within these contexts.

In the early years of CCSSM, a large task will be leading all types of teachers to an understanding of exactly what mathematical modeling is. Henry O. Pollak's introductory essay in this *Handbook* should help begin to forge this understanding. To further its development, future and current teachers should analyze the progression of each of the modules with particular focus on how students are led to think. This progression closely replicates the processes a working mathematician would use to model. Once the thinking process is understood, an understanding of modeling as a whole will begin to blossom.

The modules in the *Handbook* were written to be accessible to most students. Every student is unique and it is reasonable to try to adapt the modules to the needs of different students. Adaptation is another task that can be undertaken in teacher education and professional development programs. Consideration of students' needs, capabilities, and interests is important and adaptation of the modules in this *Handbook* is encouraged, given that the modeling process – from variable identification to model revision or refinement and reporting the results – is maintained.

A prepared teacher is one who, among other things, anticipates how students will respond to questions and tasks. Another possible task that can be undertaken in teacher education and professional development is trying to anticipate how students will answer the questions posed in the modules, what questions will cause trouble, and how to respond to these. A prepared teacher also will try to determine what to do to help students persevere in developing the model and, if necessary, what extra information can be provided to a student without "giving away" the solution. Determining other mathematically valid types of models besides those presented in the "Possible Solutions" section is also helpful. A task such as this is one all teachers should learn to undertake before teaching a particular lesson.

An interesting teacher education or professional development task would be to determine where the use of these lessons can be taught in an interdisciplinary context. Several modules can be adapted easily for use in science classrooms; some could even be used in the context of social studies, for instance. The act of developing and teaching interdisciplinary lessons using these modules should help both students and teachers understand that a person who is capable of – or at least understands – mathematical modeling is an informed citizen. This is an important lesson to be learned for anyone.

There are various mathematical topics covered within the *Handbook* that may be unfamiliar to teachers as several of them are not frequently taught even in typical undergraduate mathematics courses. This is particularly true of those topics involving discrete mathematics. The topics covered in the *Handbook* all have the "typical" mathematics at their core – number, algebra, geometry, trigonometry, and statistics – but they also frequently involve mathematics not typically seen in high school curricula. It is well-within reason for teacher education and professional development programs to engage in some "content preparation", such as presentations or short courses on the areas of mathematics that are not typically covered in many teacher preparation curricula that will allow teachers to become more familiar and comfortable with the mathematics employed in the *Handbook*.

A final suggestion for professional development tasks related to the use of this *Handbook* is to determine the best way to assess students, by both formative and summative means. The act of monitoring and evaluating students' cognitive processes is much more difficult than the act monitoring and evaluating their fluency with or acquisition of facts and procedures. Mathematical modeling is both a procedure and a cognitive process, so its evaluation is "tricky". Those engaged in teacher education and professional development programs are encouraged to devise creative and novel ways to assess the modeling activities included in this *Handbook*.

All of the activities listed above would generally be addressed during the course of a lesson study. Practicing teachers might find lesson study to be a valuable professional development activity related to mathematical modeling, and one that can be undertaken without the need to employ outside resources. Lesson study is a common activity in Japanese schools and it involves several teachers working collaboratively on a single lesson or activity in order to understand how to teach it best. While the whole process of lesson study will not be addressed here, it is recommended that teachers work together to develop plans for exactly how to facilitate the teaching of the modeling activities included in this *Handbook*. Making use of one's colleagues may prove to be the most important and helpful lesson to be learned from professional development activities.

This set of tasks is certainly not exhaustive, nor do we claim that all the suggested tasks are necessary. We do hope, however, that this *Handbook* provides a valuable and enjoyable resource for teacher education and professional development activities related to CCSSM modeling.

FOR THE BIRDS

Teacher's Guide — Getting Started

Heather Gould
Stone Ridge, NY

Purpose

In this two-day lesson, students are challenged to consider the different physical factors that affect real-world models. Students are asked to find out how long it will take a birdfeeder — with a constant stream of birds feeding at it — to empty completely.

To begin, explain that the students will be watching over a neighbor's home. This neighbor is an ornithologist (a scientist that studies birds) with a birdfeeder to be looked after. Humans can't come around too often because it will frighten the birds, but they also can't come around too infrequently because the birds will leave if the feeder frequently is empty. The students need to find out when to come back and fill the feeder to ensure that the neighbor and the birds are all happy.

Prerequisites

Students need to be very strong with algebra as there is a heavy reliance on equation manipulation in the lesson.

Materials

Required: (For a physical model) Cardboard box, rice (or sand), cylindrical plastic bottle (a Starbucks Ethos Water bottle, for example), scissors, stopwatches or timers.
Suggested: Graphing paper or a graphing utility.
Optional: None.

Worksheet 1 Guide

The first three pages constitute the first day's work. Students are given the opportunity to explore a physical model of a birdfeeder using a cylindrical, plastic bottle as the feeder and rice as the feed. Make sure the bottle is perfectly or very nearly cylindrical. Use scissors to cut "feed holes" (approximately 1 cm in diameter) in the appropriate spots, as indicated in the lesson. Cover the holes so no rice falls out until the experiment is ready to begin (a few students "plugging up" the holes with their fingers is sufficient). Hold the model feeder over a cardboard box so the rice doesn't make a mess. Use stopwatches or other timers to keep track of the total time it takes to empty as well as each of the time periods elapsed at each of the mathematically important moments.

Worksheet 2 Guide

The fourth and fifth pages of the lesson constitute the second day's work. Students need to find out how to model various different situations; they'll learn that each one has a mathematical tie-in to the birdfeeder problems. It turns out that the mathematical model they created for the birdfeeder is sufficient to solve each problem, but this is not obvious until connections are made as to how the problems are related mathematically.

CCSSM Addressed

N-Q.1: Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.
N-Q.2: Define appropriate quantities for the purpose of descriptive modeling.
A-CED.4: Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations.

FOR THE BIRDS

Student Name:____ Date:___________________

Your neighbor, an ornithologist, has to leave for the weekend to do a research study. She has asked you to make sure her birdfeeder always has food in it so that the birds keep coming back throughout the day. Refilling too seldom will cause the birds to look elsewhere for food; refilling too much will scare off the birds.

Leading Question

How often should you feed the birds so they keep coming back?

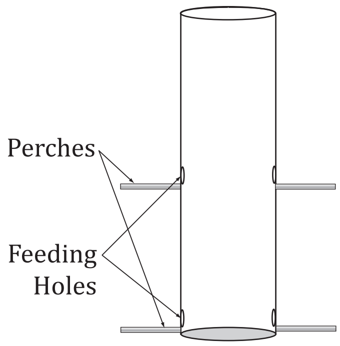
FOR THE BIRDS

Student Name:______________________________________ Date:__________________

1. Your neighbor told you that it's important not to fill the feeder too often or to fill the feeder too seldom, so how can you determine how often to fill it?

What's mathematically important about how the birdfeeder empties? Are there any important variables?

2. When you go over first thing in the morning, the birdfeeder — which has 4 holes, one pair near the bottom and another pair about halfway up (shown in the picture) — is nearly full. You check back 45 minutes later and it's about half full. When do you expect it to empty again?

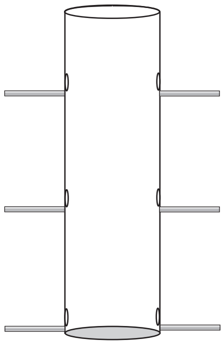


3. You come back 45 minutes later and it's still not nearly empty. Why is that? The birds are still coming by consistently to eat, so they still are hungry. When should you expect the feeder to be nearly empty and ready for you to fill it again?

4. Describe a method for calculating when the birdfeeder should be empty. Use mathematical notation, if you can.

FOR THE BIRDS

Student Name:_______________________________________ Date:__________________



You did so well taking care of your neighbor's birdfeeder that she recommended you for a weekend job watching over one of her colleague's birdfeeders. This bird-feeder has 6 feeding holes, with pairs equally spaced as shown in the picture.

5. The first morning you get there, you notice that the feeder is about 2/3 full. You wait a while and notice that it takes about 30 minutes before the feeder is about 1/3 full. How long will it take before you need to refill the feeder? How long will it take for the feeder to need to be refilled after that?

6. Build a mock birdfeeder like the one above to test your answers from question 5 above. Use a clear, cylindrical container as the birdfeeder and rice as the food. How well did your mathematical model agree with your physical model?

How should you track your findings? Are there certain important events?

7. Write a mathematical description of how to determine how quickly the birdfeeder will empty.

8. Can you *generalize* the description above? Are your answers from questions 4 and 7 similar? How so?

FOR THE BIRDS

Student Name:____ Date:__________________

9. You and 3 of your friends are making crafts for a charity sale. All of you work on Saturday and make 180 in all. On Sunday, only 2 of you can work. How many can you expect to have ready for the sale on Monday morning?

10. There is another charity sale on Saturday. You will make a new type of craft this time. You plan your schedules so that on Monday, 5 of you work; 4 work on Tuesday; 3 work on Wednesday; 2 work on Thursday; and only you make the new craft on Friday. There are 360 crafts done by the end of Tuesday. How many crafts do you expect will be done for the sale?

11. Describe, using words and mathematical notation, how you obtained your answers.

12. Are the birdfeeder problems related to the craft problems? If so, describe the relationship. Is the mathematics involved similar? Why or why not?

FOR THE BIRDS

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13. You are starting a weekend landscaping business. After the first day, you only finished 25% of the weekend's work. How many friends do you need to hire for tomorrow to help you make sure all the work gets done on time?

14. How is question 13 above similar to the birdfeeder and craft problems? How is it different? What mathematical ideas, if any, are similar? Did you use similar methods?

15. What other types of problems use methods similar to those used above? Make up and solve a problem that uses those methods.

16. What are the types of units used in the problems above? If you know the unit needed in the answer of a problem, can that help you determine how to solve it? Explain.

FOR THE BIRDS

Teacher's Guide — Possible Solutions

The solutions shown represent only some possible solution methods. Please evaluate students' solution methods on the basis of mathematical validity.

1. Important variables to consider are how quickly a portion empties, if birds will always be feeding (the lesson assumes they will, given that they are not frightened by a human tending the feeder too often or frustrated from finding too little food), and how many feeding holes there are and where they're located. The latter two variables often are overlooked.
2. One half of the birdfeeder empties in 45 minutes when the birds are able to access 4 feeding holes. After the halfway point, they are only able to access 2 feeding holes, thereby halving their rate. It takes 45+2(45) = 45+90 = 135 minutes = 2 hours, 15 minutes to empty completely. (Often, incorrect answers occur because many people don't consider the different rates.)
3. See answer 2 above.
4. Let F = one feeder, r = the rate at which the feeder empties (the unit is feeders/minute), and t = the time it takes, in minutes. Then $F = rt$ is satisfied if the rate is always constant. The challenge is that the rate changes at the halfway point. So $F = r_1t_1 + r_2t_2$. The initial situation gives $(1/2)F = r_1 \cdot (45)$. Thus, $r_1 = 1/90$. Since the rate slows based on the number of feeding holes available, $r_2 = (1/2)r_1 = (1/2)(1/90) = 1/180$. Then the following is satisfied:
   $1 = (1/90) \cdot 45 + (1/180) \cdot t_2$
   $1 = (1/2) + (1/180)t_2$
   $(1/2) = (1/180)t_2$
   $90 = t_2$
   The birdfeeder empties after $t_1 + t_2$ minutes, which is 135 minutes, or 2 hours and 15 minutes.
5. $F = r_1t_1 + r_2t_2 + r_3t_3$; $t_2 = 30$; $r_2 = 2r_3$; $r_1 = 3r_3$. Also, $(1/3)F = r_2 \cdot (30)$, so $r_2 = 1/90$. Combine these as above to get that $r_3 = 1/180$ and $t_3 = 60$. Finally, $r_1 = 1/60$, $t_1 = 20$. The total time is 110 minutes, or 1 hour and 50 minutes.
6. An accurate physical model will have few differences from the mathematical model.
7. See answer 5 above.
8. See answer 5 above.
9. If 4 people can make 180, then 2 people can make (2/4) as many crafts, or 90. Then the total number of crafts ready by Monday is 270. Mathematically, Crafts = Rate · People. This can be modified as in question 4.
10. There are 9 people each completing a workday Monday and Tuesday and they make a total of 360 crafts. Rearrange the formula to get the rate. Rate = crafts/workdays completed, so rate = 360/9 = 40 crafts/workday. So by the end of the week, 15 workdays will be completed in all. Thus, crafts = 40 (crafts/workday) · 15 workdays = 600 crafts.
11. See answer 10 above.
12. Both depend heavily on rates.
13. Rate = (1/4)(total job/person). Thus, (3/4)(total job) = (1/4)(total job/person) · 3 people. 3 people are needed.
14. This uses different rates, but all rely heavily on rate issues.
15. Answers will vary. Distance/rate/time problems, $d = rt$, are very common.
16. The unit needed can help with the rearrangement of the necessary formula and can help sort out the "direction" of the problem.

FOR THE BIRDS

Teacher's Guide — Extending the Model

If you plot your data in question 2 to how full the bird feeder is as a function of time, you have three points: at time 0, it is full ($y = 1$); at 45 minutes, it is half full ($y = 1/2 = 0.5$); and your students probably discovered that it would be empty at 135 minutes ($y = 0$). So they have three points: (0, 1); (45, 0.5); and (135,0).

What do you think happens *between* these points? You expect the birds to eat pretty steadily! So you connect (0, 1) and (45, 0.5) by a straight-line segment, and then (45, 0.5) and (135, 0) also by a straight-line segment. You have a function that is defined *piecewise.* So what would you expect to be the level of the bird feeder to have been at 18 minutes? Probably 0.8. What about at 1 hour and at 2 hours?

Suppose you want the upper part of the feeder to empty in the *same* time as it took the lower part. How can you get it to do that, with the same number of birds involved in each part? One way is to put the upper perches closer to the top! Where should you put them? You should put them 1/3 of the way down, or you could fail to fill the bird feeder completely when you start. Neither the birds nor the scientists would like that. You can now play with different vertical distances among the rows of perches, and see what variety of patterns you can get.

You have an interesting new question first: when do you think the bird feeder was originally filled? Proceeding as before you will again get a function defined-piecewise, but this time it will consist of *three* pieces. Why?

Something more should be said about piecewise-defined functions. Such functions are seen much more often in modeling the outside world than is generally realized. Here are 3 more examples.

(i) Post office functions. The simplest example is the postage for a letter as a function of its weight. Highly variable from year-to-year. Other rules, dealing with postage for packages, are more complicated.

(ii) There was an ad for the price of turkeys at a supermarket the week before Thanksgiving. It said something like 89 cents a pound for birds under 8 pounds, 69 cents a pound between 8 and 14 pounds, and 49 cents a pound above 14 pounds. What could you buy for 7 dollars? 8? 9? In the real world, you may not have all these choices. If you wait too long, you have to settle for whatever size is left.

(iii) Look at the rpm of an automobile engine as the car starts and accelerates to cruising speed. When you shift from 1st to 2nd, you get onto a different curve and it happens again on the shift from 2nd to high. When shown this function, many students, even those in engineering schools, have trouble understanding what it represents. Jeff Griffiths from Cardiff, Wales was the source of this observation.

Some of these functions are discontinuous, while others have discontinuous first derivatives. They are all defined piecewise, and they all model real situations.

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